ON THE USE OF SPHERICAL MEANS IN HYDRODYNAMICS. APPLICATION TO THE STUDY OF THE INCEPTION OF TURBULENCE

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ABSTRACT

In Part I Continuous Medium is defined and the Angular Momentum Theorem is applied to a spherical volume of the medium. Applied to perfect fuids this yields classical results. Applied to solids and to viscous fluids it yields maximum radii of stability.

In Part II a number of physical assumptions are made about the inception and the nature of turbulent flow and various formulas are obtained.

In Part III spherical means, considered as functions of the radius of the sphere, are expanded into power series. These expansions are used to analyse the limitations of the use of spherical means in hydrodynamics.

Part I: Continuous medium

1. Definition of a continuous medium

We shall assume that the medium under consideration satisfies a number of conditions, which will be taken as defining a Continuous Medium:

A) At any moment t the medium fills a three dimensional open volume.

B) At any point $P(x, y, z)$ of this volume and at the moment t the medium has a density $\rho(x, y, z, t)$ which has continuous partial derivatives.

C) At any point $P(x, y, z)$ of this volume and at the moment t the medium has a velocity $\mathbf{v}(x, y, t)$ which has continuous first and second partial derivatives (actually the existence of $\partial^2 \mathbf{v}/\partial t^2$ is not required).

D) Let a portion of the medium at the moment t fill a volume $d\tau$ and let the same portion at the moment t' fill a volume $(d\tau)'$. Let the densities of the medium at the two moments be respectively ρ and ρ', then $\rho d\tau = \rho'(d\tau')$. Hence

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$$(1) \qquad \frac{d}{dt}(\rho d\tau) = 0, \ \left(\frac{d}{dt} \text{ denotes the hydrodynamical derivative}\right),$$

which is the equation of continuity.

We note that, with the above definitions, the flow of two fluids of different densities with a surface of separation, any flow with shock waves, any turbulent flow or any flow with a boundary are not considered here to be flows of a continuous medium.

2. External and internal forces acting on a continuous medium

Let us at the moment t consider a closed surface σ completely fllled with the matter of the medium. The external forces acting on the matter inside σ are of two kinds:

1) Volume forces acting on the matter within each element of volume $d\tau$ inside.

2) Surface forces acting through the elements of surface $d\sigma$ of σ.

Furthermore we limit ourselves to the study of systems in which all internal forces appear as torques of zero moment and can be neglected in the Angular Momentum Theorem which will be the only dynamical theorem used. (Except in Part III).

3. The angular momentum theorem

Consider the portion of the continuous medium in a volume τ inside a closed surface σ. Let $d\tau$ be an element of the volume τ. Denote by $\rho(x,y,z,t)$, $\boldsymbol{v}(x,y,z,t)$ and $\boldsymbol{a}(x,y,z,t)$ respectively the density, the velocity and the acceleration of the matter in $d\tau$. We note that

$$\boldsymbol{a} = d\boldsymbol{v}/dt = \partial\boldsymbol{v}/\partial t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}.$$

Let $\rho \boldsymbol{f} d\tau$ be the outside volume force acting on the matter in $d\tau$, and let $\boldsymbol{F}d\sigma$ be the outside surface force acting through $d\sigma$. Furthermore denote by $\boldsymbol{r}$ the position vector of points in or on σ, that is the position vector of volume element $d\tau$ or of surface elements $d\sigma$ relatively to a *fixed* origin O.

Then the Angular Momentum Theorem, applied to the matter within yields

$$\iint_\sigma (\boldsymbol{r}\times\boldsymbol{F})d\sigma = \iiint_\tau (\boldsymbol{r}\times\boldsymbol{f})\rho d\tau = \frac{d}{dt}\iiint_\tau (\boldsymbol{r}\times\boldsymbol{v})\rho d\tau.$$

Differentiating under the integral sign and using the equation of continuity (1), the fact that $dr/dt = \boldsymbol{v}$ and collecting the volume integrals this formula becomes

$$(2)\qquad \iint_{\sigma} (\boldsymbol{r} \times \boldsymbol{F}) d\sigma = \iiint_{\tau} [\boldsymbol{r} \times (\boldsymbol{a} - \boldsymbol{f})] \rho d\tau,$$

and will be used in this form. Formula (2) constitutes the Angular Momentum Theorem.

4. Special case of the angular momentum theorem

We shall need here the special case of the Angular Momentum Theorem in which it is applied to the matter inside a spherical surface Σ and the origin 0 of the position vectors $\boldsymbol{r}$ is the fixed point which, at the moment t, coincides with the center of Σ. Here only the tangential components of the surface forces $\boldsymbol{F}d\Sigma$ contribute to the moments $(\boldsymbol{r} \times \boldsymbol{F})d\sigma$. We shall denote these tangential components by $\boldsymbol{T}d\Sigma$ (they are shears). Finally we shall denote by S the spherical volume inside Σ.

With these notations, the last formula (2) of §3 yields the following theorem.

THEOREM 1. *Consider a continuous medium of density $\rho(x,y,z,t)$. Inside that medium consider a spherical surface Σ of radius b, enclosing the spherical volume S. Let $\boldsymbol{r}$ be the position vector of the surface element $d\Sigma$ of Σ or of the volume element dS of S relative to the center of Σ. Let $\boldsymbol{T}d\Sigma$ be the shear applied through $d\Sigma$ to the matter in S by the outside medium and let $\rho \boldsymbol{f}(x,y,z,t)d\tau$ be the extremal volume force applied to the matter in the volume element dS. Then, if $\boldsymbol{a}$ is the acceleration of the matter in dS:*

$$(3)\qquad \iint_{\Sigma} (\boldsymbol{r} \times \boldsymbol{T}) d\Sigma = \frac{4\pi}{15} b^5 \overline{\operatorname{rot} \rho(\boldsymbol{a} - \boldsymbol{f})},$$

where $\overline{\operatorname{rot} \rho(\boldsymbol{a} - \boldsymbol{f})}$ is the weighted average of $\operatorname{rot} \rho(\boldsymbol{a} - \boldsymbol{f})$ *in S given by*

$$(4)\qquad \iiint_{S} \frac{b^2 - r^2}{2} \operatorname{rot} \rho(\boldsymbol{a} - \boldsymbol{f}) dS = \frac{4\pi}{15} b^5 \overline{\operatorname{rot} \rho(\boldsymbol{a} - \boldsymbol{f})}.$$

PROOF. The last Eq. (2) of §3 can here be written

$$\iint_{\Sigma} (\boldsymbol{r} \times \boldsymbol{T}) d\Sigma = \iiint_{S} [\boldsymbol{r} \times (\boldsymbol{a} - \boldsymbol{f})] \rho dS,$$

and it remains to transform the volume integral. We have

$$\iint_S\int[\boldsymbol{r}\times(\boldsymbol{a}-\boldsymbol{f})]\rho dS = \int_{\beta=0}^{\beta=b} d\beta\iint_{\Sigma_\beta}[\boldsymbol{r}\times(\boldsymbol{a}-\boldsymbol{f})]\rho d\Sigma_\beta,$$

where Σ_β is the spherical surface of center 0 and radius β. Putting $\boldsymbol{r} = \beta\boldsymbol{n}$ where $\boldsymbol{n}$ is the unit vector normal to $d\Sigma_\beta$ and denoting by S_β the interior of Σ_β we have, by a classical result

$$\iint_{\Sigma_\beta}[\boldsymbol{r}\times(\boldsymbol{a}-\boldsymbol{f})]\rho d\Sigma_\beta = \beta\iint_{\Sigma_\beta}\boldsymbol{n}\times(\boldsymbol{a}-\boldsymbol{f})\rho d\Sigma_\beta = \beta\iiint_{S_\beta}\operatorname{rot}\rho(\boldsymbol{a}-\boldsymbol{f})dS_\beta,$$

so that

$$\iint_S\int[\boldsymbol{r}\times(\boldsymbol{a}-\boldsymbol{f})]\rho dS = \int_{\beta=0}^{\beta=b}\beta d\beta\iiint_S\operatorname{rot}\rho(\boldsymbol{a}-\boldsymbol{f})dS_\beta = \iiint_S\frac{b^2-r^2}{2}\times\operatorname{rot}\rho(\boldsymbol{a}-\boldsymbol{f})dS.$$

We thus have

$$\iint_\Sigma(\boldsymbol{r}\times T)d\Sigma = \iiint_S\frac{b^2-r^2}{2}\operatorname{rot}\rho(\boldsymbol{a}-\boldsymbol{f})dS,$$

and the main theorem follows because

$$\iiint_S\frac{b^2-r^2}{2}dS = \frac{4\pi}{15}b^5.$$

5. Application to solids

In a solid $\operatorname{rot}\boldsymbol{a}$ is constant in space. Indeed if $\boldsymbol{\omega}$ is the angular velocity of the solid at the time t, then:

$$\operatorname{rot}\boldsymbol{a} = 2\frac{d\boldsymbol{\omega}}{dt}. \tag{5}$$

This is due to the fact (which can be checked analytically) that in a solid

$$\operatorname{rot}\frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\operatorname{rot}\boldsymbol{v},$$

and

$$\operatorname{rot}\boldsymbol{v} = 2\boldsymbol{\omega}.$$

Therefore when $\boldsymbol{f}$ is the force of gravity, for a homogeneous solid, our Theorem I becomes the well known

THEOREM II (Homogeneous solid medium). *If the medium is solid and homogeneous of constant density ρ then*

$$\iint_{\Sigma} (\boldsymbol{r} \times \boldsymbol{T}) d\Sigma = \frac{8\pi}{15} b^5 \frac{d\boldsymbol{\omega}}{dt}. \tag{6}$$

We agree to call $\frac{1}{2}\operatorname{rot}\boldsymbol{a}$ the "gyro-acceleration" of the medium, the movement being non-gyrational or gyrational according to whether this acceleration is zero or not. (Note that in a solid the gyro-acceleration coincides with the acceleration of the angular velocity $d\boldsymbol{\omega}/dt$). Theorem II shows that the value of the shears in a gyrating solid augment with its gyro-acceleration. If we now assume that the solid will crack when the shears become too large, we arrive at the conception of the maximum gyro-acceleration that a given solid can support without breaking. More precisely let A be the largest admissible value of $|\boldsymbol{T}|$ and let us compute then the largest possible value of $\left|\iint_{\Sigma}(\boldsymbol{r} \times \boldsymbol{T})d\Sigma\right|$. This will be reached if the vectors $\boldsymbol{T}$ all have the maximum length A and are all parallel to the same plane. An easy calculation shows that then:

$$\left|\iint_{\Sigma} (\boldsymbol{r} \times \boldsymbol{T}) d\Sigma\right| = \pi^2 b^3 A. \tag{7}$$

If, therefore, $(8\pi/15) b^5 \rho \left|d\boldsymbol{\omega}/dt\right| > \pi^2 b^3 A$ then there must exist on Σ points where the shear is greater than A and the solid will break. Thus the solid certainly cannot support a gyro-acceleration greater than: $\left|d\boldsymbol{\omega}/dt\right|_{max} = (15\pi/a)(A/\rho b^2)$. At lower gyro-accelerations the shears can become larger than A on some portions of Σ and the solid may start cracking. (Later we shall suppose however that in this case the shear gets redistributed over Σ until the limit is reached simultaneously on the whole surface).

THEOREM III. *Let b be the radius of the largest sphere that can be in scribed in a gyrating homogeneous solid. Then there exists a maximum gyro-acceleration which cannot be surpassed without breaking the solid. This maximum gyro-acceleration is*

$$\left|\frac{d\boldsymbol{\omega}}{dt}\right|_{\max} = \frac{15\pi}{8} \frac{A}{\rho b^2}. \tag{8}$$

Alternatively, a homogeneous solid gyrating with the gyro-acceleration $d\boldsymbol{\omega}/dt$ cannot contain without breaking, a sphere of radius greater than

(9) $$b_{max} = \frac{\left(\frac{15\pi}{8}\frac{A}{\rho}\right)^{\frac{1}{2}}}{\left(\frac{d\omega}{dt}\right)^{\frac{1}{2}}}.$$

6. Application to perfect fluids

In a perfect fluid $\boldsymbol{T} = 0$ and Theorem I yields easily

THEOREM IV (Perfect fluids). *In a non-viscous medium where all shears are zero*

(10) $$\operatorname{rot}\rho(\boldsymbol{a} - \boldsymbol{f}) = 0.$$

Three special cases are of interest:

(A) *Incompressible perfect fluid*

Here ρ is constant. If $\boldsymbol{f}$ is the constant field of gravity, $\operatorname{rot}\boldsymbol{f} = 0$ and Theorem IV reduces to

THEOREM V. *In an incompressible perfect fluid*

(11) $$\operatorname{rot}\boldsymbol{a} = 0.$$

(B) *Two-dimensional horizontal flow of a perfect fluid.*

Vector $\boldsymbol{a}$ is now horizontal while $\boldsymbol{f}$ is vertical. Writing that the vertical component of $\operatorname{rot}\rho(\boldsymbol{a} - \boldsymbol{f})$ is zero, we find in this case

THEOREM VI. *In a two-dimensional horizontal flow of a perfect fluid*

(12) $$\operatorname{rot}(\rho\boldsymbol{a}) = 0.$$

(C) *Two dimensional horizontal irrotational flow of a perfect fluid*:

In this case $\operatorname{rot}\boldsymbol{v} = 0$, which implies $\operatorname{rot}\boldsymbol{a} = 0$ (this can be checked analytically). Remembering that $\operatorname{rot}(\rho\boldsymbol{a}) = \rho\operatorname{rot}\boldsymbol{a} - \boldsymbol{a}\times\operatorname{grad}\rho$ and applying Eq. (12) we find

(13) $$\boldsymbol{a}\times\operatorname{grad}\rho = 0.$$

which yields

THEOREM VII. *In a two-dimensional irrotational flow of a perfect fluid in a horizontal plane the aceleration is parallel to the gradient of the density.*

7. Application to viscous continuous media

In the general case (including that of viscous flow) $\boldsymbol{T}$ is not zero nor is $\mathrm{rot}\,\rho(\boldsymbol{a}-\boldsymbol{f})$ constant. The condition that $|\boldsymbol{T}|$ should not exceed a limit A on Σ does not here necessarily lead to the existence of a maximum for the radius b of Σ because, if $\mathrm{rot}\,\rho(\boldsymbol{a}-\boldsymbol{f})$ changes rapidly in absolute value or direction and A is large enough, examples can be constructed to show that however large the radius b of the sphere Σ, the quantity $4\pi/15\ b^5\,\overline{\mathrm{rot}\,\rho(\boldsymbol{a}-\boldsymbol{f})}$ (the weighted average over the inside of Σ) is always smaller than $\pi^2 b^3 A$. However, if $\mathrm{rot}\,\rho(\boldsymbol{a}-\boldsymbol{f})$ is not too variable and A not too big such a maximum radius b_{max} does exist and is given, just like in the case of a solid by

THEOREM VIII. *The maxinum possible radius of a sphere of viscous fluid in laminar flow is*

$$b_{max} = \left(\frac{15\pi}{4}A\right)^{1/2} \left[\overline{(\mathrm{rot}\,\rho(\boldsymbol{a}-\boldsymbol{f})}\right]^{-\frac{1}{2}}. \tag{14}$$

It would be contrary to our conception of matter to admit the possibility of the existence of limitless shears within a material medium. If A is the greatest value attainable by shears within the medium, it then follows that if there exists a sphere of radius b_{max} completely inside the medium, then the portion of medium within that sphere will be sheared off the rest of the medium and will start gyrating independently.

In laminar viscous flow the value of b_{max} in (14) can be calculated from the Navier-Stokes equations. As an example consider the case of a liquid of constant density ρ and constant viscosity μ. The Navier-Stokes equation is, in this case:

$$\rho(\boldsymbol{a}-\boldsymbol{f}) = -\mathrm{grad}\,p + \mu\Delta\boldsymbol{v},$$

so that, using the classical result that $\Delta\boldsymbol{v} = \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{v}$ in incompressible fluids, we have

$$\mathrm{rot}\,\rho(\boldsymbol{a}-\boldsymbol{f}) = \mu\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{v},$$

whence

THEOREM IX *In a Navier-Stokes flow of an incompressible fluid, of constant viscosity the maximum possible radius of a sphere of fluid is*

$$b_{max} = \left(\frac{15\pi}{4}\,\frac{A}{\mu}\right)^{\frac{1}{2}} \left[\overline{\mathrm{rot}\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{v}}\right]^{-\frac{1}{2}}, \tag{15}$$

the average being again taken inside the sphere of radius b_{max}.

In a fluid medium the shearing off of independently gyrating drops leads to a new regime of flow, known as turbulent flow, which will be described in the following paragraphs.

Here it may be said that $R = b_{max} \cdot \left|\mathrm{rot\,rot\,rot}\,\boldsymbol{v}\right|^{\frac{1}{2}}$ is, because of (15), a kind of Reynolds number for the inception of turbulence.

Part II. Turbulence

8. Phenomenological decription

We have shown that our Theorem I together with the assumption of the non-existence of infinitely great shear within a medium lead us to predict the shearing off of finite spherical portions of the medium, breaking up its laminar flow. We claim that this is the inception of turbulence.

Once turbulence has set in, this flow should consist of large numbers of independently spinning drops surrounded by flowing fluid. Each drop is acted upon by two torques (we are not here concerned with forces acting upon the drop whose effect is merely to displace its center). There is, first of all, friction between the surface of the drop and the outer fluid. The friction creates a torque tending to make the drop rotate (and therefore also to gyrate) like the surrounding fluid. Secondly, there is a torque due to the inertia of the drop tending to preserve the angular velocity of the drop both in direction and in magnitude. As a result, the drop adopts some regime of gyration. There should exist two further effects: the outside fluid, when forced to flow around the gyrating drops will develop highly curved trajectories giving rise to large values of $\left|\mathrm{rot}\,\boldsymbol{a}\right|$ so that new drops will be sheared off and, on the other hand, the old drops will be subjected to centrifugal forces which will make them bulge at the equator and so break up. Finally, an exchange of matter is possible through the outer surface of the drops. There is therefore a constant process of formation of new drops and of breaking up of old ones.

It is interesting to get estimates of the average radius of the drops, their angular velocities and their duration.

We shall here attempt to answer the first two of these questions. The answers are at best tentative and claim no vigor. Even so they necessitate additional assumptions of a physical nature.

9. The maximum shear assumption

The first assumption is needed to determine the average radius of the drops. We state the assumption in the following precise form

ASSUMPTION. *There exists a quantity A, having the dimensions of a shear, such that if the moment of the shears acting on a spherical surface Σ is so large that it cannot be reached by any distribution of shears which are everywhere on Σ not larger than A, then the medium inside Σ will be sheared off the rest of the medium and will start gyrating as a separate drop.*

To clarify the nature of this assumption consider again a solid medium. Let there be a solid body with a spherical cavity and in this cavity a solid sphere of the same material as the rest of the body but of a slightly smaller diameter than the cavity. Suppose the sphere be attached to the body by a spherical shell of glue which cements it on all sides to the cavity. If now the body is gyrated, the sphere will be gyrated along with it. However, the glue will be subjected to shears. These shears will finally crack the glue and liberate the sphere to gyrate independently inside its cavity. Let A be the maximum shear that the glue can support without cracking. We assume that if a shear larger than A should develop at some point in the glue, the excess will be distributed over the spherical shell and that the glue will only break if no distribution of shears exists in which the shears are everywhere not greater than A. Finally the "glue" can be the material of the solid body itself. It is claimed here that a similar situation obtains within a fluid medium.

A remark has to be made: Our assumption that the shear within a fluid cannot surpass the value A will modify the Navier-Stokes equation for large shears, that is just at the passage from laminar to turbulent flow. This modification is not attempted here.

10. The radius of the drops

The effect of our Assumption is to show that the radius of the drops at the inception of turbulent flow is precisely the radius b_{max} given by formulas (14) and (15) (the latter, only if we assume that the Navier-Stokes equation applies almost up to the shearing off of the drop).

This leads us to the following rule for the inception of turbulence

RULE.

A) If the geometrical dimensions of a laminar flow are such that no sphere of radius b_{max} given by (14) or (15) can be completely immersed in the flowing fluid, then the flow will not become turbulent.

B) If these dimensions are such that in some region spheres of radius b_{max} are immersed in the fluid, the laminar flow will be unstable and turbulence will set in.

Once turbulence has set in one can no longer speak of a flow velocity $\mathbf{v}$ but only of the average flow velocity $\mathbf{u}$ (for each gyrating drop $\mathbf{u}$ can be taken as the velocity of its center). If the duration of a drop is much larger than the period of the fluctuations of $\mathbf{v}$ it seems plausible to assume that the average radius b_{max} of the independently gyrating drops in turbulent flow upon their formation is obtained from Eq. (14) giving b_{max} by replacing there $\mathbf{v}$ by $\mathbf{u}$.

However once turbulence has been established, each drop has its own regime of flow. The flow within the drop is laminar and can be studied. We do not attempt this study here. If the drop had been a homogeneous solid animated with the angular velocity $\boldsymbol{\omega}$ we would have, $\mathbf{v}$ being now the velocity of the fluid within the drop (assuming that $\operatorname{rot} \mathbf{f} = 0$)

$$\operatorname{rot} \rho\left(\frac{d\mathbf{v}}{dt} - \mathbf{f}\right) = \rho \operatorname{rot} \frac{d\mathbf{v}}{dt} = \rho \frac{d}{dt} \operatorname{rot} \mathbf{v} = 2\rho \frac{d\boldsymbol{\omega}}{dt}.$$

Using Theorem VIII we now have

THEOREM X. *The maximum radius b_{dr} of an independently gyrating drop animated with the angular velocity $\boldsymbol{\omega}$ is*

$$b_{dr} = C \left[\left|\frac{d\boldsymbol{\omega}}{dt}\right|\right]^{-\frac{1}{2}}, \tag{16}$$

where C is given, in the hypothesis that the fluid in the drop moves like a homogeneous solid, by

$$C = \frac{15\pi}{8} \frac{A}{\rho}. \tag{17}$$

We shall assume that for actual drops Theorem X still holds with $\boldsymbol{\omega}$ representing the angular velocity of the portions of the fluids situated on the equator of the surface of the drop, C being now an experimental constant. This constant will, of course, no longer be given by (17) but by a different formula to be deduced from the study of the flow within the drop.

11. Angular velocity of the drops

It has been explained above that the motion of the drop around its center is brought about by two torques the one due to friction and the other to inertia. Assuming again for the moment, that the drop moves like a homogeneous solid sphere of angular velocity $\boldsymbol{\omega}$, we find that the average angular velocity of the drop relative to the surrounding fluid is

$$\boldsymbol{\Omega} = \boldsymbol{\omega} - \tfrac{1}{2}\operatorname{rot} \boldsymbol{u}. \tag{18}$$

If we assume that the torque T_{dr} due to friction is proportional to the surface and to some power ν of the velocity of the fluid of the drop on Σ relative to the outside fluid, we find that this torque is:

$$T_{f_r} = K b_{dr}^{\nu+3} \cdot |\Omega|^{\nu-1} \boldsymbol{\Omega}\,, \tag{19}$$

where K is a numerical constant and b_{dr} is the radius of the drop given by (16). The torque T_{in} due to inertia is similarly

$$T_{in} = K' \rho b_{dr}^{5} \cdot \frac{d\boldsymbol{\omega}}{dt}, \tag{20}$$

Where K' is another constant. The equation of the motion is

$$T_{fr} + T_{in} = 0. \tag{21}$$

We deduce from (21) that $|T_{fr}| = |T_{in}|$ so that

$$K b_{dr}^{\nu+3} \cdot |\boldsymbol{\Omega}|^{\nu} = K' \rho b_{dr}^{5} \left| \frac{d\boldsymbol{\omega}}{dt} \right|. \tag{22}$$

Eliminating b_{dr} between (16) and (22) we find

$$|\boldsymbol{\Omega}| = \left(\frac{K'\rho}{K} C^{2-\nu}\right)^{1/\nu} \left(\left|\frac{d\boldsymbol{\omega}}{dt}\right|\right)^{\frac{1}{2}}. \tag{23}$$

Using the values of b_{dr} and of $|\boldsymbol{\Omega}|$ from (16) and from (23) in (19) and (20), the equation of motion becomes, after simplification

$$\boldsymbol{\Omega} + \gamma \frac{\dfrac{d\boldsymbol{\omega}}{dt}}{\left(\dfrac{d\boldsymbol{\omega}}{dt}\right)^{\frac{1}{2}}} = 0, \tag{24}$$

where γ is a positive dimensionless constant depending on K, K', ρ, C and ν. Namely

$$\gamma = \left(\frac{K'\rho}{K}\right)^{1/\nu} C^{((2-\nu)/\nu)}. \tag{25}$$

Replacing $\boldsymbol{\Omega}$ from (18) and making the same reservations as in the remark following Theorem X, we now have

THEOREM XI. *Let* $\boldsymbol{\omega}$ *be the angular velocity at the equator of an independently gyrating drop. Then* $\boldsymbol{\omega}$ *satisfies the differential equation*

$$\boldsymbol{W} - \tfrac{1}{2}\operatorname{rot}\boldsymbol{u} + \gamma \frac{\dfrac{d\boldsymbol{\omega}}{dt}}{\left(\left|\dfrac{d\boldsymbol{\omega}}{dt}\right|\right)^{\frac{1}{2}}} = 0, \tag{26}$$

where $\boldsymbol{u}$ is the average velocity of the fluid outside the drop and γ is a positive dimensionless experimental constant.

Part III. The limitations of the method of spherical means

12. It is of interest to note that though we have limited ourselves only to the Equation of Continuity and to the Angular Momentum Theorem applied to spherical volumes, we were still able to deduce a large body of classical hydrodynamical results and also, by considering finite spheres, we deduced the idea of the maximum radius (our Theorem IX). However it is equally important to understand the limitations inherent in the use of spherical means. For this purpose we need some further mathematical results.

We shall need the following theorem (which is of independent interest and can be used to compute various spherical means)

THEOREM XII *Let S be a sphere of radius b and $\boldsymbol{a}$ a vector having $2n$ continuous partial derivatives. Then*

$$\iiint_S \boldsymbol{a}\,dS = \sum_{k=0}^{k=n-1} \frac{8\pi(k+1)}{(2k+3)!} \cdot b^{2k+3} \cdot (\Delta^k \boldsymbol{a})_0 + \boldsymbol{R}_n, \tag{27}$$

with

$$\boldsymbol{R}_n = \iiint \frac{(b-r)^{2n}}{(2n+1)!}\,\frac{2nb+r}{r}(\Delta^n \boldsymbol{a})dS = \frac{8\pi(n+1)}{(2n+3)!} b^{2n+3}\overline{(\Delta^n \boldsymbol{a})}, \tag{28}$$

$\overline{(\Delta^n \boldsymbol{a})}$ *being a weighted average of $\Delta^n \boldsymbol{a}$ in S.*

And

$$\text{(29)} \qquad \iiint_S (\boldsymbol{r} \times \boldsymbol{a}) dS = \sum_{k=0}^{k=n+1} \frac{16\pi(k+1)(k+2)}{(2k+5)!} \cdot b^{2k+5} (\operatorname{rot} \Delta^k \boldsymbol{a})_0 + \boldsymbol{R}_n',$$

with

$$\text{(30)} \qquad \boldsymbol{R}_n' = \iiint_S \frac{(b-r)^{2n+1}}{(2n+3)!} \frac{3r^2 + 3(2n+1)br + 4n(n+1)b^2}{r} (\operatorname{rot} \Delta^n \boldsymbol{a}) dS$$

$$= \frac{16\pi(n+1)(n+2)}{(2n+5)!} b^{2n+5} \overline{(\operatorname{rot} \Delta^n \boldsymbol{a})}$$

$\overline{(\operatorname{rot} \Delta^n \boldsymbol{n})}$ *being a weighted average of* $\operatorname{rot} \Delta^n \boldsymbol{a}$ *in* S. *Here* $\boldsymbol{r}$ *is the position vector relative to the center of the sphere and* $(\Delta^k \boldsymbol{a})_0$ *and* $(\operatorname{rot} \Delta^k \boldsymbol{a})_0$ *are values of these functions at the center. Also* $r = |\boldsymbol{r}|$.

PROOF. The proof is by induction. For $n = 0$, (27) becomes an identity and (28) holds because we have earlier established that

$$\iiint_S (\boldsymbol{r} \times \boldsymbol{a}) dS = \iiint_S \frac{b^2 - r^2}{2} \operatorname{rot} \boldsymbol{a}\, dS.$$

It remains to prove that

$$\text{(31)} \qquad \iiint_S \frac{(b-r)^{2k}}{(2k+1)!} \frac{r+2kb}{r} \Delta^k \boldsymbol{a} \cdot dS = \frac{8\pi(k+1)}{(2k+3)!} b^{2k+3} (\Delta^k \boldsymbol{a})_0 +$$

$$+ \iiint_S \frac{(b-r)^{2k+2}}{(2k+3)!} \frac{r+2(k+1)b}{r} \Delta^{k+1} \boldsymbol{a}\, dS,$$

and that

$$\text{(32)} \qquad \iiint_S \frac{(b-r)^{2k+1}}{(2k+3)!} \frac{3r^2 + 3(2k+1)br + 4k(k+1)b^2}{r} \operatorname{rot} \Delta^k \boldsymbol{a}\, dS$$

$$= \frac{16\pi(k+1)(k+2)}{(2k+5)!} b^{2k+5} (\operatorname{rot} \Delta^k \boldsymbol{a})_0 +$$

$$+ \iiint_S \frac{(b-r)^{2k+3}}{(2k+5)!} \frac{3r^2 + 3(2k+3)br + 4(k+1)(k+2)b^2}{r} \operatorname{rot} \Delta^{k+1} \boldsymbol{a} dS.$$

These are however particular forms of the following more general identity which holds whenever $h(r)$ is defined and twice continuously derivable for $0 \leqq r \leqq b$ and $\boldsymbol{u}$ is a vector with continuous second derivatives

$$(33)\qquad \iiint\limits_S \frac{2h(r)-4(b-r)h'(r)+(b-r)^2h''(r)}{r}\boldsymbol{u}dS = 4\pi b^2h(0)(\boldsymbol{u})_0 + + \iiint\limits_S \frac{(b-r)^2}{r}h(r)\Delta\boldsymbol{u}dS.$$

Indeed (31) is obtained from (33) by taking $\boldsymbol{u} = \Delta^k a$ and:

$$h(r) = \frac{(b-r)^{2k}}{(2k+3)!}[r+2(k+1)b],$$

while (32) is obtained by taking $\boldsymbol{u} = \mathrm{rot}\Delta^k a$ and

$$h(r) = \frac{(b-r)^{2k+1}}{(2k+5)!}[3r^2+3(2k+3)br+4(k+1)(k+2)b^2].$$

To prove (33) it suffices to prove an analogous identity for each component of $\boldsymbol{u}$. We therefore prove the following lemma

LEMMA. *Let $h(r)$ be defined and twice continuously differentiable for $0 \leqq r \leqq b$, and let $m(\boldsymbol{r})$ be a scalar function defined and twice continuously differentiable for $0 \leqq r \leqq b$ where $r = |\boldsymbol{r}|$, then:*

$$(34)\qquad \iiint\limits_S \frac{2h(r)-4(b-r)h'(r)+(b-r)^2h''(r)}{r}m\,dS = 4\pi b^2h(o)(m)_0 + \iiint\limits_S \frac{(b-r)^2}{r}h(r)\Delta m dS,$$

the integral being taken over a sphere S of radius b.

PROOF OF THE LEMMA. Making use of the identity:

$$k\Delta m = k\,\mathrm{div}\,\mathrm{grad}\,m = \mathrm{div}(k\,\mathrm{grad}\,m) - \mathrm{grad}\,k\cdot\mathrm{grad}\,m,$$

and of the fact that the following improper integrals are all convergent, we have, denoting by dQ the element of the solid angle

$$\iiint\limits_S \frac{(b-r)^2}{r}h(r)\Delta m dS = \iiint\limits_S div\left[\frac{(b-r)^2}{r}h(r)\,\mathrm{grad}\,m\right]dS - \iiint\limits_S \mathrm{grad}\left[\frac{(b-r)^2}{r}h(r)\right]\cdot\mathrm{grad}\,m dS$$

$$= -\iiint\limits_S \frac{-(b^2-r^2)h(r)+r(b-r)^2h'(r)}{r^2}\frac{\partial m}{\partial r}dS$$

$$= - \iint dQ \int_{r=0}^{r=b} [-(b^2-r^2)h(r)+r(b-r)^2h'(r)] \frac{\partial m}{\partial r} dr =$$

$$= - \iint dQ[b^2h(o)(m)_0 -$$

$$- \int_{r=0}^{r=b} \{2rh(r) - 4r(b-r)h'(r) + r(b-r)^2h''(r)\}m\,dr] =$$

$$= -4\pi b^2 h(o)(m)_0 \quad + \iint_S \frac{2h(r)-4(b-r)h'(r) \,-(b-r)^2h''(r)}{r} \times m\,dS,$$

which proves the Lemma and also Theorem XII.

13. Limitations of spherical means

Returning to surface stresses $\boldsymbol{F}$ (instead of their tangential components $\boldsymbol{T}$) we can write the Momentum Theorem for the sphere as

$$\iint_\Sigma \boldsymbol{F} d\Sigma = \iiint_S \rho(\boldsymbol{a} - \boldsymbol{f}) dS,$$

and the Angular Momentum Theorem as

$$\iint (\boldsymbol{r} \times \boldsymbol{F}) d\Sigma = \iiint_S \boldsymbol{r} \times \rho(\boldsymbol{a} - \boldsymbol{f}) dS.$$

Using Theorem XII they become

$$\iint_\Sigma \boldsymbol{F} d\Sigma = \sum_{k=0}^{k=n-1} \frac{8\pi(k+1)}{(2k+3)!} b^{2k+3} [\Delta^k \rho(\boldsymbol{a} - \boldsymbol{f})]_0 + \boldsymbol{R}_n \tag{35}$$

and

$$\iint_\Sigma (\boldsymbol{r} \times \boldsymbol{F}) d\Sigma = \sum_{k=0}^{k=n-1} \frac{16\pi(k+1)(k+2)}{(2k+5)!} b^{2k+5} [\operatorname{rot} \Delta^k \rho(\boldsymbol{a} - \boldsymbol{f})]_0 + \boldsymbol{R}'_n. \tag{36}$$

We now want to develop the left hand sides of (35) and (36) in powers of b. To do this we have to use the fact that $\boldsymbol{F}$ is a tensor. We apply the Momentum Theorem to an elementary tetrahedron [3]. We then find that the force applied to the element of surface $d\Sigma$ normal to the unit vector

$$\boldsymbol{n} = \alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma\boldsymbol{k},$$

is

$$\boldsymbol{F}d\Sigma = (\alpha\boldsymbol{F}_x + \beta\boldsymbol{F}_y + \gamma\boldsymbol{F}_z)d\Sigma,$$

where $\boldsymbol{F}_x$, $\boldsymbol{F}_y$ and $\boldsymbol{F}_z$ are three vectors. At a point of Σ where the position vector is $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ we therefore have

$$\begin{aligned}\boldsymbol{F}d\Sigma &= \frac{1}{b}[x\,\boldsymbol{F}_x(x,y,z) + y\boldsymbol{F}_y(x,y,z) + z\boldsymbol{F}_z(x,y,z)d\Sigma \\ &= \frac{1}{b}[\{x(\boldsymbol{F}_x)_0 + y(\boldsymbol{F}_y)_0 + z(\boldsymbol{F}_z)_0\} \\ &\quad + \left\{x^2\left(\frac{\partial \boldsymbol{F}_x}{\partial x}\right)_0 + y^2\left(\frac{\partial \boldsymbol{F}_y}{\partial y}\right)_0 + z^2\left(\frac{\partial \boldsymbol{F}_z}{\partial z}\right)_0 + xy\left(\frac{\partial \boldsymbol{F}_x}{\partial y}\right)_0 + xy\left(\frac{\partial \boldsymbol{F}_y}{\partial x}\right)_0 + \cdots\right\} \\ &\quad + \cdots]d\Sigma.\end{aligned}$$

Integrated over Σ, only the terms containing solely even powers of x, y or z contribute significantly. Of the terms written out, only those in x^2, y^2 and z^2 contribute. But

$$\frac{1}{b}\iint_\Sigma x^2 d\Sigma = \frac{1}{b}\iint_\Sigma y^2 d\Sigma = \frac{1}{b}\iint_\Sigma z^2 d\Sigma = \frac{4\pi}{3}b^3.$$

Comparing with the term in b^3 on the right hand side of (35) we get, at the point 0

$$\frac{\partial \boldsymbol{F}_x}{\partial x} + \frac{\partial \boldsymbol{F}_y}{\partial y} + \frac{\partial \boldsymbol{F}_z}{\partial z} = \rho(\boldsymbol{a} - \boldsymbol{f}). \tag{37}$$

This is the classical fundamental equation of hydrodynamics. Nothing new is added by equating the coefficients of higher powers of b. Indeed one simply gets the identities obtained by taking Δ of both sides of (37).

Applying the same procedure to Equation (36) we find

$$(\boldsymbol{r} \times \boldsymbol{F})d\Sigma = \frac{1}{b}[\{x^2(\boldsymbol{i} \times \boldsymbol{F}_x) + y^2(\boldsymbol{j} \times \boldsymbol{F}_y) + z^2(\boldsymbol{k} \times \boldsymbol{F}_z) + xy(\boldsymbol{i} \times \boldsymbol{F}_y) + \cdots\} + \cdots]d\Sigma,$$

all terms of third and higher degrees in x, y and z being omitted. Equating the coefficients of b^3 on both sides of (36) we find that

$$(\boldsymbol{i} \times \boldsymbol{F}_x) + (\boldsymbol{j} \times \boldsymbol{F}_y) + (\boldsymbol{k} \times \boldsymbol{F}_z) = 0. \tag{38}$$

Denoting the components of the vector $\boldsymbol{F}_x$ by F_{xx}, F_{xy} and F_{xz} respectively and similarly for $\boldsymbol{F}_y$ and $\boldsymbol{F}_z$, the equation becomes equivalent to the system

$$F_{yz} = F_{zy}, \quad F_{zx} = F_{xz}, \quad F_{xy} = F_{yz}. \tag{39}$$

This time something more can be gained by equating the coefficients of higher powers of b in (36). Indeed, equating the coefficients of b^5 we find, by somewhat arduous computation and making use of (39) that:

$$\operatorname{rot}\left[\frac{\partial \boldsymbol{F}_x}{\partial x} + \frac{\partial \boldsymbol{F}_y}{\partial y} + \frac{\partial \boldsymbol{F}_z}{\partial z}\right] = \operatorname{rot}\rho(\boldsymbol{a} - \boldsymbol{f}), \tag{40}$$

which yields somewhat less than our Equation (37). Higher powers of b yield nothing new.

To summarize, we see that the use of spherical volumes in the Momentum Theorem does not yield the tensorial nature of the surface forces, but if this is assumed, it does yield the fundamental Equation (37). The same assumption with the Angular Momentum Theorem leads to the fact that the stress tensor is symmetrical as shown by (39). For all this there is no necessity to use finite spherical volumes. However, the Angular Momentum Theorem alone applied to a finite spherical volume and coupled with the assumption of the tensorial character of $\boldsymbol{F}$, does lead to Equation (40) which is a weaker form of the fundamental Equation (37).

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